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Pairs of partitions without repeated odd parts

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ABSTRACT

We prove two identities related to overpartition pairs. One of them gives a generalization of an identity due to Lovejoy, which was used in a joint work by Bringmann and Lovejoy to derive congruences for overpartition pairs. We apply our two identities on pairs of partitions where each partition has no repeated odd parts. We also present three partition statistics that give combinatorial explanations to a congruence modulo 3 satisfied by these partition pairs.

1. Introduction

A partition π of n is a sequence of nonincreasing integers, the sum of which equals n, while an overpartition λ of n is a partition of n where the first occurrence of a number may be overlined. In [1], Bringmann and Lovejoy considered *overpartition pairs* (λ, μ) of n; an overpartition pair is a pair of overpartitions with the sum of all parts equal to n. Define $\overline{pp}(n)$ as the number of overpartition pairs of n. Then from [1], the function $\overline{pp}$ has the generating function

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}.$$

In the equation above and for the rest of this article, we use the notation

$$(x_1, x_2, \dots, x_k; q)_m := \prod_{n=0}^{m-1} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n),$$

$$(x_1, x_2, \dots, x_k; q)_\infty := \prod_{n=0}^{\infty} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n),$$

$$[x_1, x_2, \dots, x_k; q]_\infty := (x_1, q/x_1, x_2, q/x_2, \dots, x_k, q/x_k; q)_\infty,$$

and we require $|q| < 1$ for absolute convergence.

A rank statistic for $\overline{pp}$ is given in [1]. By giving an appropriate ordering to the parts of an overpartition pair (λ, μ), Bringmann and Lovejoy defined the rank of an overpartition pair (λ, μ) as

$$\ell((\lambda, \mu)) - n(\lambda) - \overline{n}(\mu) - \chi((\lambda, \mu)),$$

where $\ell((\lambda, \mu))$, $n(\lambda)$, and $\overline{n}(\mu)$ denote the largest part of (λ, μ), largest part of λ, and largest overlined part of μ, respectively, and $\chi((\lambda, \mu))$ is defined to be 1 if the largest part of (λ, μ) is non-overlined in μ and 0 otherwise. Define

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also $\overline{NN}(m, n)$ as the number of overpartition pairs of n whose rank is m, and $\overline{NN}(r, t, n)$ as the number of overpartition pairs of n whose rank is congruent to r modulo t. It is shown in [1, Proposition 2.1] that $\overline{NN}$ has the generating function

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{NN}(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{(-1;q)_n^2 q^n}{(zq, q/z; q)_n},$$

and from [1, Theorem 1.2]

$$\overline{NN}(r, 3, 3n+2) = \frac{\overline{pp}(3n+2)}{3}, \tag{1.1}$$

which gives a combinatorial interpretation of the congruence [1, Corollary 1.3]

$$\overline{pp}(3n+2) \equiv 0 (\mathrm{mod}\ 3). \tag{1.2}$$

Results on rank differences modulo 3 and 4 are given in [1]; in particular the generating function for the rank difference modulo 3 [1, Theorem 1.4] is

$$4 + \sum_{n=1}^{\infty}\left(\overline{NN}(0,3,n) - \overline{NN}(1,3,n)\right) q^n = 4\frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^2}{(q^3;q^3)_\infty^2 (q^{18};q^{18})_\infty} + 4q\frac{(q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty (q^9;q^9)_\infty}.$$

The proof of (1.1) and results on rank differences are dependent on the identity [2, Eq. (1.11)]

$$\frac{4}{(1+z)(1+z^{-1})} + \sum_{n=1}^{\infty}\frac{(-1;q)_n^2 q^n}{(zq, q/z; q)_n} = \frac{4(-q;q)_\infty^2}{(1+z)(1+z^{-1})(zq, q/z; q)_\infty}, \tag{1.3}$$

which relates the generating function for ranks of overpartition pairs to a generating function for the *cranks of overpartition pairs* $\frac{(-q;q)_\infty^2}{(zq,q/z;q)_\infty}$. The result (1.1) is an analog of the combinatorial interpretation of two of the Ramanujan congruences for the partition function $p(n)$,

$$p(5n+4) \equiv 0 (\mathrm{mod}\ 5),$$
$$p(7n+5) \equiv 0 (\mathrm{mod}\ 7).$$

Define

$$\sum_{n=0}^{\infty} t(n)q^n := \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Then $t(n)$ counts the number of partitions of n without repeated odd parts. In [3], Lovejoy and Osburn evaluated rank differences for $t(n)$. We shall count partition pairs (λ, μ) of n, where each partition, λ and μ, does not have repeated odd parts, and the sum of all the parts of λ and μ is n. We shall simply call them *partition pairs of n without repeated odd parts*. Let $tt(n)$ denote the number of such partition pairs of n. It is clear that $tt(n)$ has the generating function

$$\sum_{n=0}^{\infty} tt(n)q^n = \frac{(-q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}.$$

We state a result analogous to (1.2), which first appeared in [4, Corollary 2.1].

Theorem 1.1. *We have* $tt(3n+2) \equiv 0 (\mathrm{mod}\ 3)$.

We remark that from [5, Theorem 1.6 and Section 3], we see that the function tt has congruences

$$tt(25n+14) \equiv tt(25n+24) \equiv 0 (\mathrm{mod}\ 5).$$

Recently, Toh [6] gave an extensive discussion on pairs of various partition functions satisfying congruences modulo 3.

The first objective of this article is to present the following two identities, the first of which gives a generalization of (1.3). We prove these two identities in Section 3.

Theorem 1.2. *We have*

$$\sum_{n=0}^{\infty}\frac{(x, 1/x; q)_n q^n}{(zq, q/z; q)_n} = \frac{(1-z)^2}{(1-z/x)(1-xz)} + \frac{z(x, 1/x; q)_\infty}{(1-z/x)(1-xz)(zq, q/z; q)_\infty}, \tag{1.4}$$

$$\sum_{n=0}^{\infty}\frac{(x, q/x; q)_n q^n}{(z, q/z; q)_{n+1}} = \frac{1}{x(1-z/x)(1-q/(xz))} + \frac{[x;q]_\infty}{z(1-x/z)(1-q/(xz))[z;q]_\infty}. \tag{1.5}$$

In [7], Lovejoy and Mallet defined two very general generating functions for overpartition pairs, and showed how one can obtain many q-series identities related to overpartitions from their result. We remark that (1.4) and (1.5) do not follow from results in that paper.

Combining the summand for $n = 0$ on the left side of (1.4) with the first product on the right side, we obtain

$$z\frac{(1-x)(1-1/x)}{(1-z/x)(1-xz)} + \sum_{n=1}^{\infty}\frac{(x,1/x;q)_n q^n}{(qz,q/z;q)_n} = \frac{z(x,1/x;q)_\infty}{(1-z/x)(1-xz)(zq,q/z;q)_\infty}.$$

From this identity, setting $x = -1$ we recover (1.3). Similarly, dividing both sides by $(1-x)(1-1/x)$ and setting $x = 1$, we recover another q-identity of Lovejoy, which we state in the following corollary.

Corollary 1.3 (*Lovejoy [2, Theorem 1.4]*). *We have*

$$\frac{4z}{(1+z)^2} + \sum_{n=1}^{\infty}\frac{(-1;q)_n^2 q^n}{(zq,q/z;q)_n} = \frac{4z(q^2;q^2)_\infty}{(1+z)(q;q^2)_\infty(q,zq,q/z;q)_\infty},$$

$$\frac{z}{(1-z)^2} + \sum_{n=1}^{\infty}\frac{(q;q)_{n-1}^2 q^n}{(zq,q/z;q)_n} = \frac{z(q;q)_\infty^2}{(1-z)^2(zq,q/z;q)_\infty}.$$

Similarly, we consider combining the summand for $n = 0$ on the left side of (1.5) and the first product on the right side. Setting $x = -1$ and dividing both sides by $(x-1)$ and setting $x = 1$ gives the following two identities, respectively.

Corollary 1.4. *We have*

$$\frac{2}{(1-z^2)(1-q^2/z^2)} + \sum_{n=1}^{\infty}\frac{(-1,-q;q)_n q^n}{(z,q/z;q)_{n+1}} = \frac{2(-q;q)_\infty^2}{(1+z)(1+q/z)[z;q]_\infty},$$

$$\frac{(1-q)}{(1-z)^2(1-q/z)^2} - \sum_{n=1}^{\infty}\frac{(q;q)_{n-1}(q;q)_n q^n}{(z,q/z;q)_{n+1}} = \frac{(q;q)_\infty^2}{(1-z)(1-q/z)[z;q]_\infty}.$$

We could also construct different specializations of (1.4) and (1.5) in order to get results analogous to Corollary 1.3.

Corollary 1.5. *We have*

$$\frac{(1+qz)(1+q/z)}{(1+q)}\sum_{n=0}^{\infty}\frac{(-q,-1/q;q^2)_n q^{2n}}{(q^2z,q^2/z;q^2)_n} = \frac{q(1-z)^2}{z(1+q)} + \frac{(-q;q^2)_\infty^2}{(zq^2,q^2/z;q^2)_\infty}, \tag{1.6}$$

$$\frac{z(1+q/z)^2}{(1-q^2/z)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n^2 q^{2n}}{(zq^2,q^4/z;q^2)_n} = z - 1 + \frac{(-q;q^2)_\infty^2}{(zq^2,q^2/z;q^2)_\infty}. \tag{1.7}$$

Identity (1.6) is obtained by replacing q by q^2 and setting $x = -q$ in (1.4), multiplying both sides of the resultant identity by $\frac{(1+zq)(1+q/z)}{(1+q)}$ and simplifying. Identity (1.7) is obtained by replacing q by q^2 and setting $x = -q$ in (1.5), multiplying both sides of the resultant identity by $z(1+q/z)^2(1-z)$ and simplifying.

Setting $z = 1$ in (1.6) and (1.7), respectively, we recover on the right sides the generating function for tt, $\frac{(-q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}$. These suggest defining rank type functions

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} TT_1(m,n)z^m q^n := \frac{(1+qz)(1+q/z)}{(1+q)}\sum_{n=0}^{\infty}\frac{(-q,-1/q;q^2)_n q^{2n}}{(q^2z,q^2/z;q^2)_n}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} TT_2(m,n)z^m q^n := \frac{z(1+q/z)^2}{(1-q^2/z)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n^2 q^{2n}}{(zq^2,q^4/z;q^2)_n}.$$

It is obvious from the definition that $TT_1(m,n) = TT_1(-m,n)$ while from (1.7), it is obvious that $TT_2(m,n) = TT_2(-m,n)$ except for $(m,n) = (\pm 1, 0)$.

To describe what TT_1 and TT_2 count, we first define the *rank* of a partition pair without repeated odd parts. We order the parts in a partition pair (λ,μ) by stipulating that for a number k,

$$k_\lambda > k_\mu,$$

where the subscript indicates to which of the two partitions the part belongs. Similarly to [1], we use the notation $l(\cdot)$, $n(\cdot)$, and $n_o(\cdot)$ for the largest part, the number of parts, and the number of odd parts of an object, respectively.

Definition 1.6. The rank of a partition pair without repeated odd parts, (λ, μ), is

$$\left\lceil \frac{l((\lambda, \mu))}{2} \right\rceil - n(\lambda) - n_o(\mu),$$

where $\lceil \cdot \rceil$ is the ceiling function.

Theorem 1.7. *Let $TT(0, 0) = 1$, and for $m \neq 0$, let $TT(m, 0) = 0$. For $n \geq 1$, let $TT(m, n)$ denote the number of partition pairs of n without repeated odd parts with rank m. Then we have the following generating function:*

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} TT(m, n) z^m q^n = \frac{(-q; q^2)_\infty^2}{(zq^2, q^2/z; q^2)_\infty}.$$

We accomplish the following in Section 2. We give a proof of Theorem 1.7. Next, we apply Theorem 1.7 to give descriptions of TT_1 and TT_2. Finally, we show that all three partition statistics TT, TT_1, and TT_2 give combinatorial explanations for the congruence in Theorem 1.1.

Remark. By examining the product representation in Theorem 1.7, we define the crank of a partition pair without repeated odd parts, (λ, μ), as the number of even parts in λ minus the number of even parts in μ. This way, the rank and the crank for partition pairs without repeated odd parts have the same generating function. Unlike the ranks and cranks for the ordinary partition, in this case, the number of such partition pairs (λ, μ) of n with crank m is equal to the number of the partition pairs with rank m. It would be very interesting if one could find a bijection for this.

2. Ranks

We first give a proof of Theorem 1.7. Our proof follows the method of proof of Proposition 2.1 in [1].

Proof of Theorem 1.7. We split the partition pairs without repeated odd parts into four cases, depending on whether the largest part is even or odd and whether it is in λ or μ; then we get four series. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-q/z; q^2)_n^2 q^{2n} z^{n-1}}{(q^2/z, q^2; q^2)_n}$$

is the generating function for partition pairs without repeated odd parts whose largest part $2n$ is in λ, where the exponent of q is the number being partitioned and the exponent of z is the rank. Combining this with the three other cases gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} TT(m, n) z^m q^n &= 1 + \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_n^2 q^{2n} z^{n-1}}{(q^2/z, q^2; q^2)_n} + \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_{n-1}^2 q^{2n-1} z^{n-1}}{(q^2/z, q^2; q^2)_{n-1}} \\
&\quad + \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_n^2 q^{2n} z^n}{(q^2/z; q^2)_{n-1} (q^2; q^2)_n} + \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_{n-1} (-q/z; q^2)_n q^{2n-1} z^{n-1}}{(q^2/z, q^2; q^2)_{n-1}} \\
&= 1 + q + (1 + qz) \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_n^2 q^{2n} z^{n-1}}{(q^2/z, q^2; q^2)_n} \\
&\quad + (1 + qz) \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_{n-1} (-q/z; q^2)_n q^{2n-1} z^{n-1}}{(q^2/z; q^2)_{n-1} (q^2; q^2)_n} \\
&= 1 + q + (1 + qz) \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_{n-1} (-q/z; q^2)_n q^{2n-1} z^{n-1}}{(q^2/z; q^2)_{n-1} (q^2; q^2)_n} \times \left(1 + q \frac{1 + q^{2n-1}/z}{1 - q^{2n}/z} \right) \\
&= (1 + q) \left(1 + (1 + qz) \sum_{n=1}^{\infty} \frac{(-q/z; q^2)_{n-1} (-q/z; q^2)_n q^{2n-1} z^{n-1}}{(q^2/z; q^2)_n (q^2; q^2)_n} \right) \\
&= (1 + q) \sum_{n=0}^{\infty} \frac{\left(\frac{-1}{qz}; q^2 \right)_n (-q/z; q^2)_n q^{2n} z^n}{(q^2/z; q^2)_n (q^2; q^2)_n} \\
&= \frac{(-q; q^2)_\infty^2}{(zq^2, q^2/z; q^2)_\infty},
\end{aligned}$$

where in last equality, we invoked the q-Gauss summation [8],

$$\sum_{n \geq 0} \frac{(a, b)_n (c/ab)^n}{(c, q)_n} = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty}. \quad \square$$

By (1.6) and (1.7), we have the following corollary.

Corollary 2.1. *For $n \geq 1$, we have*

$$TT_1(m,n) = \begin{cases} TT(m,n) - (-1)^n, & \text{for } m = \pm 1; \\ TT(0,n) + 2(-1)^n, & \text{for } m = 0; \\ TT(m,n), & \text{otherwise}. \end{cases}$$

For $n \geq 1$, $TT_2(m,n) = TT(m,n)$ and counts the number of partition pairs of n without repeated odd parts with rank m.

Let $TT(r,t,n) = \sum_{m \equiv r (\text{mod } t)} TT(m,n)$, and define $TT_1(r,t,n)$ and $TT_2(r,t,n)$ similarly. Setting $z = \omega$, a cube root of unity, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{r=0}^{2} TT(r,3,n)\omega^r q^n &= \frac{(-q;q^2)_\infty^2}{(\omega q^2, q^2/\omega; q^2)_\infty} = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty}{(q^6;q^6)_\infty} = \frac{1}{(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty} q^{n^2} \\ &= \frac{1}{(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{9n^2} + \sum_{n=-\infty}^{\infty} q^{9n^2+6n+1} + \sum_{n=-\infty}^{\infty} q^{9n^2-6n+1}\right) \\ &= \frac{(-q^9,-q^9,q^{18};q^{18})_\infty}{(q^6;q^6)_\infty} + 2q\frac{(-q^3,-q^{15},q^{18};q^{18})_\infty}{(q^6;q^6)_\infty}, \end{aligned} \tag{2.1}$$

where in the second equality, we applied the Jacobi triple product identity [9, pp. 33–36],

$$\sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{2.2}$$

with $a = b = q$, and in the last equality, we applied (2.2) three times, with $a = b = -q^9$, $a = -q^3$, $b = -q^{15}$, and $a = -q^{15}$, $b = -q^3$, respectively.

For any natural number n, by examining the terms q^{3n+2} in (2.1), we find that for $n \geq 0$,

$$TT(r,3,3n+2) = \frac{tt(3n+2)}{3} \quad \text{for } 0 \leq r \leq 2.$$

By Corollary 2.1, we also have

$$TT_1(0,3,3n+2) = \frac{tt(3n+2)}{3} + 2(-1)^n,$$

$$TT_1(1,3,3n+2) = TT_1(2,3,3n+2) = \frac{tt(3n+2)}{3} - (-1)^n,$$

$$TT_2(r,3,3n+2) = \frac{tt(3n+2)}{3} \quad \text{for } 0 \leq r \leq 2.$$

Each of these gives a combinatorial interpretation of the result $tt(3n+2) \equiv 0 (\text{mod } 3)$ stated in Theorem 1.1.

3. Proof of Theorem 1.2

We first prove (1.4). We require the following lemma in our proof.

Lemma 3.1. *We have the identity*

$$\begin{aligned}&\frac{(q;q)_\infty^2}{[x;q]_\infty}\frac{xz}{(1-x)(1-z/x)(1-xz)} + \frac{(q;q)_\infty^2}{[z;q]_\infty}\frac{zx}{(1-z)(1-x/z)(1-xz)} \\ &\quad = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(k^2+5k)/2}}{(1-xq^k)(1-q^k/x)(1-q^k/z)(1-zq^k)}. \end{aligned} \tag{3.1}$$

Proof. Setting $r = 3$, $s = 4$ in [10, (2.2)], we have

$$\begin{aligned}&\frac{(a_1q, q/a_1, a_2q, q/a_2, q/a_3q, q; q)_\infty}{[b_1, b_2, b_3, b_4; q]_\infty} \\ &\quad = \frac{[a_1/b_1, a_2/b_1, a_3/b_1; q]_\infty}{[b_2/b_1, b_3/b_1, b_4/b_1; q]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_1^3 a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_1q^k)(1-b_1q^k/a_1)(1-b_1q^k/a_2)(1-b_1q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_2b_3b_4}\right)^k\end{aligned}$$

$$+\frac{[a_1/b_2, a_2/b_2, a_3/b_2; q]_\infty}{[b_1/b_2, b_3/b_2, b_4/b_2; q]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_2^3a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_2q^k)(1-b_2q^k/a_1)(1-b_2q^k/a_2)(1-b_2q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_1b_3b_4}\right)^k$$
$$+\frac{[a_1/b_3, a_2/b_3, a_3/b_3; q]_\infty}{[b_1/b_3, b_2/b_3, b_4/b_3; q]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_3^3a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_3q^k)(1-b_3q^k/a_1)(1-b_3q^k/a_2)(1-b_3q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_1b_2b_4}\right)^k$$
$$+\frac{[a_1/b_4, a_2/b_4, a_3/b_4; q]_\infty}{[b_1/b_4, b_2/b_4, b_3/b_4; q]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_4^3a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_4q^k)(1-b_4q^k/a_1)(1-b_4q^k/a_2)(1-b_4q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_1b_2b_3}\right)^k.$$

Letting $a_1 \to b_1, a_2 \to b_2, a_3 \to b_3$, we obtain

$$\begin{aligned}\frac{(q;q)_\infty^2}{(1-b_1)(1-b_2)(1-b_3)[b_4;q]_\infty} &= \frac{(q;q)_\infty^2}{[b_4/b_1;q]_\infty}\frac{b_1^2b_2^{-1}b_3^{-1}}{(1-b_1)(1-b_1/b_2)(1-b_1/b_3)}\\ &\quad+\frac{(q;q)_\infty^2}{[b_4/b_2;q]_\infty}\frac{b_2^2b_1^{-1}b_3^{-1}}{(1-b_2)(1-b_2/b_1)(1-b_2/b_3)}\\ &\quad+\frac{(q;q)_\infty^2}{[b_4/b_3;q]_\infty}\frac{b_3^2b_1^{-1}b_2^{-1}}{(1-b_3)(1-b_3/b_1)(1-b_3/b_2)}\\ &\quad+\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+7k)/2}b_4^3b_1^{-1}b_2^{-1}b_3^{-1}}{(1-b_4q^k)(1-b_4q^k/b_1)(1-b_4q^k/b_2)(1-b_4q^k/b_3)}.\end{aligned}\tag{3.2}$$

Setting $b_1 = x^2, b_2 = xz, b_3 = x/z, b_4 = xq$, we arrive at

$$\begin{aligned}\frac{(q;q)_\infty^2}{(1-x^2)(1-xz)(1-x/z)[xq;q]_\infty} &= \frac{(q;q)_\infty^2}{[q/x;q]_\infty}\frac{x^2}{(1-x^2)(1-x/z)(1-xz)}\\ &\quad+\frac{(q;q)_\infty^2}{[q/z;q]_\infty}\frac{z^3/x}{(1-xz)(1-z/x)(1-z^2)}\\ &\quad+\frac{(q;q)_\infty^2}{[zq;q]_\infty}\frac{z^{-3}x^{-1}}{(1-x/z)(1-1/(xz))(1-1/z^2)}\\ &\quad+\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+7k+6)/2}/x}{(1-xq^{k+1})(1-q^{k+1}/x)(1-q^{k+1}/z)(1-zq^{k+1})}.\end{aligned}$$

Replacing k by $k-1$ in the series on the right side, multiplying both sides by x and simplifying, we find that

$$\begin{aligned}-\frac{x^2(q;q)_\infty^2}{(1-x^2)(1-xz)(1-x/z)[x;q]_\infty} &= \frac{(q;q)_\infty^2}{[x;q]_\infty}\frac{x^3}{(1-x^2)(1-x/z)(1-xz)}\\ &\quad+\frac{(q;q)_\infty^2}{[z;q]_\infty}\frac{z^3}{(1-xz)(1-z/x)(1-z^2)}\\ &\quad-\frac{(q;q)_\infty^2}{[z;q]_\infty}\frac{z^{-2}}{(1-x/z)(1-1/(xz))(1-1/z^2)}\\ &\quad+\sum_{k=-\infty}^{\infty}\frac{(-1)^kq^{(k^2+5k)/2}}{(1-xq^k)(1-q^k/x)(1-q^k/z)(1-zq^k)}.\end{aligned}$$

Combining the product on the left side with the first product on the right side, combining the second and third products on the right side, and rearranging, we have

$$\begin{aligned}&\frac{(q;q)_\infty^2}{[x;q]_\infty}\frac{xz}{(1-x)(1-z/x)(1-xz)}+\frac{(q;q)_\infty^2}{[z;q]_\infty}\frac{zx}{(1-z)(1-x/z)(1-xz)}\\ &\quad=\sum_{k=-\infty}^{\infty}\frac{(-1)^kq^{(k^2+5k)/2}}{(1-xq^k)(1-q^k/x)(1-q^k/z)(1-zq^k)},\end{aligned}$$

and this completes the proof. □

Proof of Identity (1.4). From a limiting case of Watson's ${}_8\phi_7$ transformation [9, Eq. (7.2), p. 16],

$$\sum_{n=0}^{\infty}\frac{(aq/bc,d,e;q)_n\left(\frac{aq}{de}\right)^n}{(q,aq/b,aq/c;q)_n}=\frac{(aq/d,aq/e;q)_\infty}{(aq,aq/de;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d,e;q)_n(1-aq^{2n})(-a^2)^nq^{n(n+3)/2}}{(q,aq/b,aq/c,aq/d,aq/e;q)_n(1-a)(bcde)^n},\tag{3.3}$$

we set $a = 1$, $b = 1/z$, $c = z$, $d = x$, $e = 1/x$. Upon simplifying, we find that

$$\sum_{n=0}^{\infty} \frac{(x, 1/x; q)_n q^n}{(qz, q/z; q)_n} = \frac{(xq, q/x; q)_\infty}{(q; q)_\infty^2} \left(1 + \sum_{n=1}^{\infty} \frac{(1+q^n)(1-1/z)(1-z)(1-1/x)(1-x)(-1)^n q^{(n^2+3n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)} \right). \tag{3.4}$$

Noting that

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^{(n^2+3n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)} = \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{(n^2+5n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)},$$

we have

$$\begin{aligned} &1 + \sum_{n=1}^{\infty} \frac{(1+q^n)(1-1/z)(1-z)(1-1/x)(1-x)(-1)^n q^{(n^2+3n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)} \\ &\quad = (1-1/z)(1-z)(1-1/x)(1-x) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n^2+5n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)}, \end{aligned} \tag{3.5}$$

since the series is absolutely convergent for $|q| < 1$. Substituting (3.5) into the series in parentheses on the right side of (3.4), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(x, 1/x; q)_n q^n}{(zq, q/z; q)_n} &= \frac{(xq, q/x; q)_\infty}{(q; q)_\infty^2} (1-1/z)(1-z)(1-1/x)(1-x) \\ &\quad \times \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n^2+5n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)}. \end{aligned}$$

Finally, invoking (3.1) on the right side, we arrive at

$$\sum_{n=0}^{\infty} \frac{(x, 1/x; q)_n q^n}{(qz, q/z; q)_n} = \frac{(1-z)^2}{(1-z/x)(1-xz)} + \frac{z(x, 1/x; q)_\infty}{(1-z/x)(1-xz)(zq, q/z; q)_\infty},$$

and this completes the proof of (1.4). □

Next, we prove (1.5). As our method of proof is similar to that of (1.4), we are more brief in our proof. We require the following lemma.

Lemma 3.2. *We have*

$$\begin{aligned} &\frac{(q; q)_\infty^2}{z(1-x/z)(1-q/(xz))[z; q]_\infty} - \frac{(q; q)_\infty^2}{z(1-x/z)(1-q^2/(xz))[x; q]_\infty} \\ &\quad = \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1} q^{(k^2+7k+2)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)}. \end{aligned} \tag{3.6}$$

Identity (3.6) is obtained by substituting $b_1 = q/z^2$, $b_2 = x/z$, $b_3 = q/(xz)$, $b_4 = q/z$ in (3.2) and applying elementary manipulations. The proof is straightforward and so we omit the details.

Proof of (1.5). Setting $a = q$, $b = z$, $c = q/z$, $d = x$, $e = q/x$, in (3.3), we obtain

$$\sum_{n=0}^{\infty} \frac{(x, q/x; q)_n q^n}{(zq, q^2/z; q)_n} = \frac{(q^2/x, xq; q)_\infty}{(q^2, q; q)_\infty} \sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(z, q/z, x, q/x; q)_n (-1)^n q^{(n^2+3n)/2}}{(1-q)(q^2/z, zq, q^2/x, xq; q)_n}.$$

Therefore, dividing both sides by $(1-z)(1-q/z)$, and simplifying, we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(x, q/x; q)_n q^n}{(z, q/z; q)_{n+1}} &= \frac{(x, q/x; q)_\infty}{(q; q)_\infty^2} \sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(-1)^n q^{(n^2+3n)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)} \\ &= \frac{(x, q/x; q)_\infty}{(q; q)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{(n^2+7n+2)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)}, \end{aligned} \tag{3.7}$$

where in the last equality, we used the fact that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n^2+3n)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)} = \sum_{n=-\infty}^{-1} \frac{(-1)^{n+1} q^{(n^2+7n+2)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)}$$

and that the series is absolutely convergent for $|q| < 1$.

Finally, we invoke (3.6) for the series on the right side of (3.7). Upon simplification, we arrive at (1.5) and this completes the proof. □

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